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Extension of the discrete KP hierarchy

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Abstract

In this paper we discuss possible generalizations of the discrete KP (Toda lattice) hierarchy. As a result, we introduce the integrable hierarchy which can be considered as the proper extension of the discrete KP hierarchy. Such extension forces us to introduce an infinite number of additional multi-times $t^{(n)} \equiv (t_1^{(n)} \equiv x^{(n)}, t_2^{(n)}, \ldots)$, $n \geqslant 2$, whereas ordinary discrete KP is relevant to $t^{(1)}$. It is shown that any subsystem of extended discrete KP attached to multi-time $t^{(n)}$ is in fact equivalent to a bi-infinite sequence of continuous KP hierarchies whose Lax operators are glued together by compatible 'gauge' transformations.

This paper can be thought of as a natural continuation and generalization of our previous paper (Svinin A K (2001) *J. Phys. A: Math. Gen.* **34** 10559–68).

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1. Introduction

It is now recognized that discrete equations play an important role in integrability theory, helping in the study of discrete symmetries of some continuous models and their respective solutions (see, for example, [2, 3, 11] and references therein). The interrelation between the discrete and continuous integrable hierarchies allows one to obtain solutions to the discrete multi-matrix models [10–12].

There are studies [3] in which some integrable lattices are treated as a union of sites, each being a gauge copy of continuous multi-boson (constrained) KP hierarchies. On the whole, these works deal with generalized Toda lattices and so-called 'square-root' lattices connected with the latter by Miura transformations.

In our recent work [1], we introduced a broad class of lattices with finite number of fields satisfying a commonly used integrability criterion—having a Lax pair. Besides others, this class contains many familiar integrable models such as the Volterra lattice, its generalizations sometimes referred to as Bogoyavlenskii lattices, generalized Toda chains and others known in the literature [15]. To some extent the paper [1] can be considered as a continuation of a series of the papers [3].

This paper is a generalization of [1]. The principal result consists of constructing of extension of the well known discrete KP hierarchy. It turned out that many known integrable lattices can be interpreted as certain specializations of the extended discrete KP. This construction forces us to introduce additional multi-times $t^{(n)}$. We refer to any subsystem attached to $t^{(n)}$ with fixed $n \geqslant 1$, due to [14], as the nth discrete KP.

We show in this paper that the nth discrete KP is in fact equivalent to a bi-infinite sequence of copies of differential KP hierarchy whose Lax operators are connected to each other by compatible 'gauge' transformations. The compatibility of the latter turned out to be equivalent to equations of motion which represent the first flow in nth discrete KP.

The paper is organized as follows. After giving some notation in section 2, in section 3 we introduce and discuss the extension of the discrete KP. Section 4 is devoted to providing a relationship between nth discrete KP and sequence of differential KP.

2. The differential and discrete KP hierarchy

Let us recall some basic facts about the differential KP hierarchy in the spirit of the Sato theory [5–7]. This approach is essentially based on the calculus of the pseudo-differential operators (ΨDOs) [8]. For reasons of completeness a certain amount of notation has to be introduced.

The unknown functions (fields) depend on spatial variable $t_1 \equiv x \in \boldsymbol{R}^1$ and some evolution parameters $t_2, t_3, \ldots$. The symbols ∂ and ∂_p stand for derivation with respect to x and t_p, respectively. In this section the symbol t denotes KP multi-time, i.e. the infinite set of evolution parameters $(t_1, t_2, t_3, \ldots)$. Let R be a commutative ring consisting of all smooth functions $a = a(x)$. Then the noncommutative ring $R[\partial, \partial^{-1})$ of ΨDOs consists of all formal expressions

$$A = \sum_{i=-\infty}^{N} a_i(x)\partial^i \qquad N \in \boldsymbol{Z}$$

with coefficients in R. One says that ΨDO A is of order N. The operator $\partial : R \to R$ is entirely defined by the generalized Leibniz rule

$$\partial^i \circ a = \sum_{j=0}^{\infty} \binom{i}{j} a^{(j)}\partial^{i-j}$$

where $a^{(j)} \equiv \partial^j a$. The adjoint of A is given by

$$A^* = \sum_{i=-\infty}^{N} (-\partial)^i \circ a_i.$$

An important part of the theory deals with the decomposition of elements of $R[\partial, \partial^{-1})$ into positive (differential) and negative (integral) parts. We denote

$$A_+ = \sum_{i \geqslant 0} a_i(x)\partial^i \qquad A_- = \sum_{i \leqslant -1} a_i(x)\partial^i$$

respectively.

It is convenient to introduce a formal dressing operator $\hat{w} = 1 + \sum_{k\geqslant 1} w_k\partial^{-k}$. Then the KP hierarchy can be represented via Sato–Wilson equations

$$\partial_p\hat{w} = -(\hat{w}\partial^p\hat{w}^{-1})_-\hat{w} = (\hat{w}\partial^p\hat{w}^{-1})_+\hat{w} - \hat{w}\partial^p \qquad (1)$$

or equivalently as Lax equations

$$\partial_p \mathcal{Q} = [(\mathcal{Q}^p)_+, \mathcal{Q}] \equiv (\mathcal{Q}^p)_+\mathcal{Q} - \mathcal{Q}(\mathcal{Q}^p)_+ \tag{2}$$

on first-order ΨDO $\mathcal{Q} = \hat{w}\partial\hat{w}^{-1} = \partial + \sum_{k\geqslant 1} u_k(t)\partial^{-k}$. A very important observation is that evolution equations of the KP hierarchy are solved in terms of a single τ-function satisfying an infinite set of bilinear equations which are encoded in the fundamental bilinear identity

$$\mathrm{res}_z[\psi(t,z)\psi^*(t',z)] \equiv \frac{1}{2\pi\mathrm{i}}\oint_0 \psi(t,z)\psi^*(t',z)\,\mathrm{d}z = 0. \tag{3}$$

Recall that the formal Baker–Akhiezer wavefunction ψ and its conjugate ψ^* entering fundamental identity are related to the KP τ-function via

$$\psi(t,z) = \frac{\tau(t-[z^{-1}])}{\tau(t)}\exp(\xi(t,z)) \qquad \psi^*(t,z) = \frac{\tau(t+[z^{-1}])}{\tau(t)}\exp(-\xi(t,z))$$

with $\xi(t,z) = \sum_{p=1}^{\infty} t_p z^p$ and $[z^{-1}] = (1/z, 1/(2z^2), 1/(3z^3), \ldots)$. Then the bilinear identity (3) becomes

$$\exp\left(\sum_{p\geqslant 1} a_p D_{t_p}\right)\sum_{k=0}^{\infty} p_k(-2a)p_{k+1}(\tilde{D}_t)\tau\bullet\tau = 0 \qquad \forall a = (a_1, a_2, \ldots).$$

A few remarks are in order. For a given polynomial $p(\partial/\partial t_1, \partial/\partial t_2, \ldots)$ in $\partial/\partial t_i$, one defines

$$p(D_{t_1}, D_{t_2}, \ldots) f\bullet g = p\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots\right) f(t_1+u_1, t_2+u_2, \ldots)g(t_1-u_1, t_2-u_2, \ldots)\Big|_{u=0}.$$

In what follows $\tilde{D}_t \equiv \left(D_{t_1}, \frac{1}{2}D_{t_2}, \frac{1}{3}D_{t_3}, \ldots\right)$. It is worth also recalling the identity

$$\frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^k f(t+[u])g(t-[u])\Big|_{u=0} = p_k(\tilde{D})f\bullet g \tag{4}$$

which will be useful in what follows. Here Schur polynomials $p_k(t)$ are defined through

$$\exp\left(\sum_{p=1}^{\infty} t_p z^p\right) = \sum_{k=0}^{\infty} z^k p_k(t).$$

The discrete KP is a KP hierarchy where the linear problem gets replaced by its discrete counterpart. More exactly, equations of motion of the discrete KP are encoded by the Lax equation

$$\frac{\partial Q}{\partial t_p} = [Q_+^p, Q]$$

on difference operator $Q = \Lambda + \sum_{k\geqslant 1} a_{k-1}\Lambda^{1-k}$. The main reference in this context is the paper by Ueno and Takasaki [4]. This hierarchy, as well as a large class of its solutions, is well described in [9]. What we learn from this work (see also [10]) is that the discrete KP is tantamount to a bi-infinite sequence of differential KP copies 'glued' together by Darboux–Bäcklund (DB) transformations. This leads to certain bilinear relations connecting the consecutive KP τ-functions. It should be noted that integrable systems which are chains of infinitely many copies of KP-type differential hierarchies turn out to be useful in matrix models [10–12]. In the work [13] by Dickey, it was shown that the discrete KP hierarchy is the most natural generalization of the modified KP.

3. Extended discrete KP hierarchy

Throughout the paper, we deal with $\infty \times \infty$ matrices. Given the shift operator $\Lambda = (\delta_{i,j-1})_{i,j\in Z}$ and the 'spectral' parameter z one considers the following spaces of the difference operators (z acts as component-wise multiplication):

$$\mathcal{D}^{(n,r)} = \left\{ \sum_{-\infty \ll k < \infty} l_k z^{k(n-1)} \Lambda^{r-kn} \right\} = \mathcal{D}_-^{(n,r)} + \mathcal{D}_+^{(n,r)} \qquad n \geqslant 1 \; r \in \boldsymbol{Z}$$

with $l_k \equiv (l_k(i))_{i\in Z}$ being bi-infinite diagonal matrices. One can easily check the following properties:

$$\begin{aligned} &\mathcal{D}^{(n,r_1)} \cdot \mathcal{D}^{(n,r_2)} \subset \mathcal{D}^{(n,r_1+r_2)} \qquad && \Lambda \cdot \mathcal{D}^{(n,r)} \subset \mathcal{D}^{(n,r+1)} \\ &\mathcal{D}^{(n,r)} \cdot \Lambda \subset \mathcal{D}^{(n,r+1)} && z^{n-1}\mathcal{D}^{(n,r)} \subset \mathcal{D}^{(n,r+n)}. \end{aligned}$$

Remark 1. In the case $n = 1$, a dependence of $\mathcal{D}^{(1,r)}$ on r make no sense because $L \in \mathcal{D}^{(1,r)}$ does not depend on z.

The splitting of $\mathcal{D}^{(n,r)}$ into 'negative' and 'positive' parts is defined as follows:

$$\mathcal{D}_-^{(n,r)} = \left\{ \sum_{r-kn\leqslant -1} l_k z^{k(n-1)} \Lambda^{r-kn} \right\} \qquad \mathcal{D}_+^{(n,r)} = \left\{ \sum_{r-kn\geqslant 0} l_k z^{k(n-1)} \Lambda^{r-kn} \right\}.$$

In the following we assume that the entries of l_k may depend on multi-time $t \equiv \{t_p^{(n)}\}$. For corresponding time derivatives we use the following notation: $\partial_p^{(n)} = \partial/\partial t_p^{(n)}$ and $\partial^{(n)} = \partial/\partial x^{(n)}$, where $x^{(n)} \equiv t_1^{(n)}$.

The phase space $\mathcal{M}$ consists of the entries of diagonal matrices $w_k = (w_k(i))_{i\in Z}$, $k \geqslant 1$. For each $n \geqslant 1$, we define the 'wave' operator

$$S^{(n)} = I + \sum_{k\geqslant 1} w_k z^{k(n-1)} \Lambda^{-kn} \in I + \mathcal{D}_-^{(n,0)} \tag{5}$$

and the corresponding Lax operator

$$Q^{(n)} \equiv S^{(n)} \Lambda S^{(n)-1} = \Lambda + \sum_{k\geqslant 1} a_{k-1}^{(n)} z^{k(n-1)} \Lambda^{1-kn} \in \mathcal{D}^{(n,1)}. \tag{6}$$

It is clear that the coordinates $a_k^{(n)}$ are related to the original ones by some polynomial relations. For example, from (6) one can read off the following:

$$\begin{aligned} a_0^{(n)}(i) &= w_1(i) - w_1(i+1) \\ a_1^{(n)}(i) &= w_2(i) - w_2(i+1) + w_1(i-n+1)(w_1(i+1) - w_1(i)) \\ a_2^{(n)}(i) &= w_3(i) - w_3(i+1) + w_1(i-2n+1)(w_2(i+1) - w_2(i)) \\ &\quad + w_2(i-n+1)(w_1(i+1) - w_1(i)) \\ &\quad + w_1(i-2n+1) w_1(i-n+1)(w_1(i) - w_1(i+1)). \end{aligned}$$

Now we are in position to define the flows on $\mathcal{M}$ with respect to parameters $t_p^{(n)}$. This uses the equations of motion on the 'wave' operator (henceforth we shall employ the shorthand notation S for $S^{(n)}$ and Q for $Q^{(n)}$ whenever this will lead to confusion)

$$\begin{aligned} z^{p(n-1)} \frac{\partial S}{\partial t_p^{(n)}} &= Q_+^{pn} S - S\Lambda^{pn} \in \mathcal{D}^{(n,pn)} \\ z^{p(n-1)} \frac{\partial (S^{-1})^T}{\partial t_p^{(n)}} &= (S^{-1})^T \Lambda^{-pn} - (Q_+^{pn})^T (S^{-1})^T. \end{aligned} \tag{7}$$

Note that the first and second equations in (7) are in fact equivalent. Evolutions of S induces evolutions of Q in the form of the Lax equations

$$z^{p(n-1)}\frac{\partial Q}{\partial t_p^{(n)}} = [Q_+^{pn}, Q] \in \mathcal{D}^{(n,pn+1)}. \tag{8}$$

One can easily check that $[Q_+^{pn}, Q] = -[Q_-^{pn}, Q]$ is of the same form as the lhs of (8) and therefore (8) and equivalent equations (7) are properly defined.

Obviously, the discrete KP hierarchy can be regarded as a subsystem of (7) with respect to the infinite set of parameters $t^{(1)} = (t_1^{(1)}, t_2^{(1)}, \ldots)$. For this reason, we refer to (7) as the extended discrete KP hierarchy. The subsystem of (7) corresponding to evolution parameters $t^{(n)} = (t_1^{(n)}, t_2^{(n)}, \ldots)$ we call, following [14], the nth discrete KP hierarchy.

It is also useful to consider the evolution equations

$$z^{p(n-1)}\frac{\partial Q^r}{\partial t_p^{(n)}} = [Q_+^{pn}, Q^r] \qquad r \in \mathbf{Z} \tag{9}$$

that follow from (8). It is easy to see that the rth power of Q is of the form

$$Q^r = \Lambda^r + \sum_{k\geqslant 1} a_{k-1}^{(n,r)} z^{k(n-1)}\Lambda^{r-kn} \in \mathcal{D}^{(n,r)}$$

with diagonal matrices $a_k^{(n,r)}$ whose entries are polynomially expressed via original coordinates $w_k(i)$. For example

$$\begin{aligned}
a_0^{(n,r)}(i) &= \sum_{s=1}^{r} a_0^{(n)}(i+s-1) \qquad &&\text{for } r \geqslant 1\\
a_0^{(n,r)}(i) &= -\sum_{s=1}^{-r} a_0^{(n)}(i-s) \qquad &&\text{for } r \leqslant -1\\
a_0^{(n,0)}(i) &\equiv 0.
\end{aligned}$$

4. nth discrete KP

Define $\chi(z) = (z^i)_{i\in\mathbf{Z}}$, $\chi^*(z) = \chi(z^{-1})$ and wavevectors

$$\Psi(t,z) = W\chi(z) \qquad \Psi^*(t,z) = (W^{-1})^T\chi^*(z) \tag{10}$$

where $W \equiv S\exp\left(\sum_{p=1}^{\infty} t_p^{(n)}\Lambda^p\right)$. The discrete linear system

$$\begin{aligned}
&Q\Psi(t,z) = z\Psi(t,z) \qquad && Q^T\Psi^*(t,z) = z\Psi^*(t,z)\\
&z^{p(n-1)}\partial_p^{(n)}\Psi = Q_+^{pn}\Psi \qquad && z^{p(n-1)}\partial_p^{(n)}\Psi^* = -(Q_+^{pn})^T\Psi^*
\end{aligned} \tag{11}$$

clearly follows from (7) and (10). From (10), it follows that

$$\begin{aligned}
\Psi_i(t,z) &= z^i(1 + w_1(i)z^{-1} + w_2(i)z^{-2} + \cdots)\mathrm{e}^{\xi(t^{(n)},z)}\\
&= z^i(1 + w_1(i)\partial^{(n)-1} + w_2(i)\partial^{(n)-2} + \cdots)\mathrm{e}^{\xi(t^{(n)},z)}\\
&\equiv z^i\hat{w}_i(\partial^{(n)})\mathrm{e}^{\xi(t^{(n)},z)} \equiv z^i\psi_i(t,z).
\end{aligned}$$

Next, we show the equivalence of the nth discrete KP to the bi-infinite sequence of differential KP copies 'glued' together by compatible gauge transformations, one of which can be recognized as the DB transformation mapping $\mathcal{Q}_i \equiv \hat{w}_i\partial^{(n)}\hat{w}_i^{-1}$ to $\mathcal{Q}_{i+n} \equiv \hat{w}_{i+n}\partial^{(n)}\hat{w}_{i+n}^{-1}$.

Proposition 1. *The following three statements are equivalent:*

(i) The wavevector $\Psi(t, z)$ *satisfies the discrete linear system*

$$Q^r\Psi(t, z) = z^r\Psi(t, z) \qquad z^{n-1}\partial^{(n)}\Psi = Q_+^n\Psi \qquad r \in \boldsymbol{Z}. \tag{12}$$

(ii) The components ψ_i *of the vector* $\psi \equiv (\psi_i = z^{-i}\Psi_i)_{i\in\boldsymbol{Z}}$ *satisfy*

$$G_i^{(r)}\psi_i(t, z) = z\psi_{i+n-r}(t, z) \qquad H_i\psi_i(t, z) = z\psi_{i+n}(t, z) \tag{13}$$

with $H_i \equiv \partial^{(n)} - \sum_{s=1}^n a_0^{(n)}(i + s - 1)$ *and*

$$G_i^{(r)} \equiv H_i + a_0^{(n,r)}(i + n - r) + a_1^{(n,r)}(i + n - r)H_{i-n}^{-1} + a_2^{(n,r)}(i + n - r)H_{i-2n}^{-1}H_{i-n}^{-1} + \cdots.$$

(iii) For the sequence of $\partial^{(n)}$*-dressing operators* $\{\hat{w}_i,\ i \in \boldsymbol{Z}\}$ *the equations*

$$G_i^{(r)}\hat{w}_i = \hat{w}_{i+n-r}\partial^{(n)} \qquad H_i\hat{w}_i = \hat{w}_{i+n}\partial^{(n)} \tag{14}$$

hold.

Remark 2. Since $Q^0 = I$ and $a_k^{(n,0)}(i) = 0$, we have in this case $G_i^{(0)} = H_i$.

Proof of Proposition 1. Rewrite equations (12) in explicit form:

$$\Psi_{i+r} + a_0^{(n,r)}(i)z^{n-1}\Psi_{i+r-n} + a_1^{(n,r)}(i)z^{2(n-1)}\Psi_{i+r-2n} + \cdots = z^r\Psi_i$$

$$z^{n-1}\partial^{(n)}\Psi_i = \Psi_{i+n} + z^{n-1}\left(\sum_{s=1}^n a_0^{(n)}(i + s - 1)\right)\Psi_i.$$

In terms of wavefunctions ψ_i the latter is rewritten as

$$z\psi_{i+r} + a_0^{(n,r)}(i)\psi_{i+r-n} + \frac{1}{z}a_1^{(n,r)}(i)\psi_{i+r-2n} + \cdots = z\psi_i \tag{15}$$

$$\partial^{(n)}\psi_i = z\psi_{i+n} + \left(\sum_{s=1}^n a_0^{(n)}(i + s - 1)\right)\psi_i. \tag{16}$$

One sees that equation (16) coincides with the second one in (13). Shifting $i \to i - r + n$ in (15) and combining it with (16) one can obtain the first equation in (13). Therefore we have proved (i) $\Rightarrow$ (ii). The converse can also easily be shown by returning to the functions Ψ_i. The equivalence (ii) $\leftarrow$ (iii) follows from the representation $\psi_i(t, z) = \hat{w}_i e^{\xi(t^{(n)},z)}$. $\square$

Let us write down in explicit form the equations of motion coded in the Lax equation

$$z^{n-1}\partial^{(n)}Q^r = [Q_+^n, Q^r] \tag{17}$$

which is the consistency condition of the linear discrete system (12). We have

$$\partial^{(n)}a_k^{(n,r)}(i) = a_{k+1}^{(n,r)}(i + n) - a_{k+1}^{(n,r)}(i) + a_k^{(n,r)}(i)\left(\sum_{s=1}^n a_0^{(n)}(i + s - 1) - \sum_{s=1}^n a_0^{(n)}(i + s + r - (k + 1)n - 1)\right) \qquad k = 0, 1, \ldots. \tag{18}$$

Notice that the simplest form of these flows is in the original coordinates:

$$\partial^{(n)}w_k(i) = w_{k+1}(i + n) - w_{k+1}(i) + w_k(i)(w_1(i) - w_1(i + n)).$$

System (18) allows for obvious reductions specified by conditions $a_k^{(n,r)}(i) \equiv 0$ when $k \geqslant l$ with some $l \geqslant 1$. Reducing (18) along this line leads to a variety of l-field lattices. The interested reader can find in [1] a collection of integrable lattices known from the literature [15] which can be derived as the above-mentioned reductions of (18). Unfortunately, the case $r \leqslant -1$ in [1] was overlooked. To fill this gap, let us recollect the Belov–Chaltikian lattice [16] (here $' \equiv \partial/\partial x^{(1)}$, $a_0(i) \equiv a_0^{(1)}(i)$ and $a_1(i) \equiv a_1^{(1,-1)}(i)$):

$$\begin{aligned} a_0'(i) &= a_1(i+1) - a_1(i+2) + a_0(i)(a_0(i+1) - a_0(i-1)) \\ a_1'(i) &= a_1(i)(a_0(i) - a_0(i-3)) \end{aligned}$$

which is relevant, as can be checked, to the specialization $n = 1, r = -1, l = 2$.

Define an infinite set of ΨDOs $\{G_i^{(\ell,r)},\ i, \ell, r \in \boldsymbol{Z}\}$ by means of the following recurrence relations:

$$\begin{aligned} G_i^{(\ell+1,r)} &= G_{i+n}^{(\ell,r)} H_i & \ell &= 0, 1, 2, \ldots \\ G_i^{(\ell-1,r)} &= G_{i-n}^{(\ell,r)} H_{i-n}^{-1} & \ell &= 0, -1, -2, \ldots \end{aligned} \tag{19}$$

with

$$\begin{aligned} G_i^{(0,r)} \equiv G_{i-n}^{(r)} H_{i-n}^{-1} = 1 &+ a_0^{(n,r)}(i-r) H_{i-n}^{-1} \\ &+ a_1^{(n,r)}(i-r) H_{i-2n}^{-1} H_{i-n}^{-1} + a_2^{(n,r)}(i-r) H_{i-3n}^{-1} H_{i-2n}^{-1} H_{i-n}^{-1} + \cdots. \end{aligned}$$

It is important to observe that

$$a_\ell^{(n,r)}(i + \ell n - r) = \operatorname{res}_{\partial^{(n)}} G_i^{(\ell,r)}. \tag{20}$$

Proposition 2. *The following auxiliary equations hold:*

$$G_i^{(\ell,r)} \psi_i = z^\ell \psi_{i+\ell n - r}. \tag{21}$$

Proof. By induction. Let $\ell = 0$, then

$$G_i^{(0,r)} \psi_i = G_{i-n}^{(r)} H_{i-n}^{-1} \psi_i = z^{-1} G_{i-n}^{(r)} \psi_{i-n} = \psi_{i-r}.$$

Now suppose that (21) is true for some l, then

$$G_i^{(\ell+1,r)} \psi_i = G_{i+n}^{(\ell,r)} H_i \psi_i = z G_{i+n}^{(\ell,r)} \psi_{i+n} = z^{\ell+1} \psi_{i+(\ell+1)n-r}.$$

This proves (21) for positive integers ℓ. Similar arguments are used for negative ℓ. □

As a consequence of the proposition, we obtain $G_i^{(\ell,\ell n)} = \mathcal{Q}_i^\ell$. Note that equation (21) in equivalent form is rewritten as

$$G_i^{(\ell,r)} \hat{w}_i = \hat{w}_{i+\ell n - r} \partial^{(n)\ell}. \tag{22}$$

Proposition 3. *The relation*

$$G_{i+\ell_2 n - r_2}^{(\ell_1,r_1)} G_i^{(\ell_2,r_2)} = G_i^{(\ell_1+\ell_2, r_1+r_2)} \tag{23}$$

holds.

Proof. Taking into account (22), we obtain that left multiplication of the lhs and rhs of (23) on $\hat{w}_i$ gives the same result, namely $\hat{w}_{i+(\ell_1+\ell_2)n-r_1-r_2}(\partial^{(n)})^{\ell_1+\ell_2}$. This proves the proposition. □

As we have mentioned above, the consistency condition of the linear system (12) expressed in the form of a Lax equation reads in explicit form as the lattice (18). As a consequence of proposition 3 we obtain that this system guarantees the validity of permutation relations

$$G_{i+\ell_2 n - r_2}^{(\ell_1,r_1)} G_i^{(\ell_2,r_2)} = G_{i+\ell_4 n - r_4}^{(\ell_3,r_3)} G_i^{(\ell_4,r_4)} \tag{24}$$

with arbitrary integers $\{\ell_k, r_k\}_{k=1}^4$ such that $\ell_1 + \ell_2 = \ell_3 + \ell_4$ and $r_1 + r_2 = r_3 + r_4$. It is clear that permutation relation (24) can be extended on that with arbitrary number of cofactors. In addition, since $G_i^{(0,0)} = 1$ we have

$$G_i^{(\ell,r)^{-1}} = G_{i+\ell n - r}^{(-\ell,-r)}.$$

From the above we learn that system (18) guarantees that the set of bi-infinite sequences of ΨDOs $\{G_i^{(\ell,r)},\ i \in \boldsymbol{Z}\}$ endowed with the multiplication rule (23) bears the structure of the group isomorphic to $\boldsymbol{Z} \times \boldsymbol{Z}$.

Proposition 4. *By virtue of* (22) *and its consistency condition* (24), $\partial^{(n)}$*-Lax operators* $\mathcal{Q}_i$ *are connected with each other by invertible compatible gauge transformations*

$$\mathcal{Q}_{i+\ell n-r} = G_i^{(\ell,r)} \mathcal{Q}_i G_i^{(\ell,r)^{-1}}. \tag{25}$$

Remark 3. Since $\mathcal{Q}_i^\ell = G_i^{(\ell,\ell n)}$, the relation (25) in the case $r = \ell n$ becomes a trivial identity.

Proof of Proposition 4. Taking into account (22), we have

$$\begin{aligned}\mathcal{Q}_{i+\ell n-r} &= \hat{w}_{i+\ell n-r} \partial \hat{w}_{i+\ell n-r}^{-1} = (G_i^{(\ell,r)} \hat{w}_i \partial^{(n)-1}) \partial^{(n)} (\partial^{(n)} \hat{w}_i^{-1} G_i^{(\ell,r)-1}) \\ &= G_i^{(\ell,r)} \hat{w}_i \partial^{(n)} \hat{w}_i^{-1} G_i^{(\ell,r)-1} = G_i^{(\ell,r)} \mathcal{Q}_i G_i^{(\ell,r)-1}.\end{aligned}$$

The mapping $\mathcal{Q}_i \to \tilde{\mathcal{Q}}_i = \mathcal{Q}_{i+m}$, where $m = \ell n - r$, we denote as s_m.

Let $m_1 = \ell_1 n - r_1$ and $m_2 = \ell_2 n - r_2$. By virtue of (24), where $\ell_3 = \ell_2$, $\ell_4 = \ell_1$, $r_3 = r_2$ and $r_4 = r_1$ we get

$$\begin{aligned}\mathcal{Q}_{i+m_1+m_2} &= G_{i+m_2}^{(\ell_1,r_1)} \mathcal{Q}_{i+m_2} G_{i+m_2}^{(\ell_1,r_1)-1} \\ &= G_{i+m_2}^{(\ell_1,r_1)} G_i^{(r_2)} \mathcal{Q}_i G_i^{(\ell_2,r_2)-1} G_{i+m_2}^{(\ell_1,r_1)-1} \\ &= G_{i+m_1}^{(\ell_2,r_2)} G_i^{(\ell_1,r_1)} \mathcal{Q}_i G_i^{(\ell,r_1)-1} G_{i+m_1}^{(\ell_2,r_2)-1} \\ &= G_{i+m_1}^{(\ell_2,r_2)} \mathcal{Q}_{i+m_1} G_{i+m_1}^{(\ell_2,r_2)-1}.\end{aligned}$$

From this follows the pair-wise compatibility of transformations s_{m_1} and s_{m_2} for any integers m_1 and m_2. So we can write $s_{m_1} \circ s_{m_2} = s_{m_2} \circ s_{m_1}$. The inverse maps s_m^{-1} are well defined by the formula $\mathcal{Q}_{i-\ell n+r} = G_{i-\ell n+r}^{(\ell,r)-1} \mathcal{Q}_i G_{i-\ell n+r}^{(\ell,r)} = G_i^{(-\ell,-r)} \mathcal{Q}_i G_i^{(-\ell,-r)-1}$. □

Rewrite the second equation in (13) as $z^{n-1} H_i \Psi_i(t,z) = \Psi_{i+n}(t,z) = (\Lambda^n \Psi)_i$. From this we derive

$$\begin{aligned}z^{k(1-n)} (\Lambda^{kn} \Psi)_i &= H_{i+(k-1)n} \cdots H_{i+n} H_i \Psi_i \\ z^{k(n-1)} (\Lambda^{-kn} \Psi)_i &= H_{i-kn}^{-1} \cdots H_{i-2n}^{-1} H_{i-n}^{-1} \Psi_i.\end{aligned}$$

These relations make a one-to-one connection between difference operators

$$P = \sum_{k \in \boldsymbol{Z}} z^{k(1-n)} p_k(t) \Lambda^{kn} \in \mathcal{D}^{(n,0)}$$

and the sequences of $\partial^{(n)}$-pseudo-differential operators $\{\mathcal{P}_i,\ i \in \boldsymbol{Z}\}$ mapping the upper triangular part of the given matrix (including the main diagonal) into the differential parts of $\mathcal{P}_i$ and the lower triangular part of the matrix to the purely pseudo-differential parts. More exactly, we have $(P\Psi)_i = \mathcal{P}_i \Psi_i$, $(P_-\Psi)_i = (\mathcal{P}_i)_- \Psi_i$ and $(P_+\Psi)_i = (\mathcal{P}_i)_+ \Psi_i$, where

$$\begin{aligned}\mathcal{P}_i &= \sum_{k>0} p_{-k}(i,t) H_{i-kn}^{-1} \ldots H_{i-2n}^{-1} H_{i-n}^{-1} + \sum_{k \geqslant 0} p_k(i,t) H_{i+(k-1)n} \cdots H_{i+n} H_i \\ &= (\mathcal{P}_i)_- + (\mathcal{P}_i)_+.\end{aligned}$$

In what follows, we denote $\sigma : P \in \mathcal{D}^{(n,0)} \to \{\mathcal{P}_i,\ i \in \boldsymbol{Z}\}$. It is easy to check that

$$\sigma : z^{\ell(1-n)} \Lambda^{\ell n - r} Q^r \to \{G_i^{(\ell,r)}, i \in \boldsymbol{Z}\}. \tag{26}$$

Proposition 5. *Equations* $z^{p(n-1)} \partial_p^{(n)} \Psi = Q_+^{pn} \Psi,\ p = 2, 3, \ldots$ *are equivalent to* $\partial_p^{(n)} \psi_i = (\mathcal{Q}_i^p)_+ \psi_i,\ p = 2, 3, \ldots.$

Proof. Setting $r = \ell n$ in (26) gives

$$\sigma : z^{p(1-n)} Q^{pn} \to \{\mathcal{Q}_i^p, i \in \boldsymbol{Z}\}.$$

Taking this into account, we have

$$z^i \partial_p^{(n)} \psi_i = \partial_p^{(n)} \Psi_i = z^{p(1-n)} (Q_+^{pn} \Psi)_i = (\mathcal{Q}_i^p)_+ \Psi_i = z^i (\mathcal{Q}_i^p)_+ \psi_i$$

which proves the proposition. □

We learn from this proposition that the nth discrete KP is in fact equivalent to a bi-infinite sequence of differential KP hierarchies whose evolution equations can be written as the Sato–Wilson equation

$$\partial_p^{(n)} \hat{w}_i = (\mathcal{Q}_i^p)_+ \hat{w}_i - \hat{w}_i \partial^{(n)p} \tag{27}$$

where $\hat{w}_i$ are connected by relations (14), or equivalently as the Lax equation

$$\partial_p^{(n)} \mathcal{Q}_i = [(\mathcal{Q}_i^p)_+, \mathcal{Q}_i] \tag{28}$$

where $\mathcal{Q}_i$ are connected by the gauge transformations (25).

Let us establish equations treating $G_i^{(r)}$-evolutions with respect to KP flows. Differentiating the lhs and rhs of (14), by virtue of (27), formally leads to evolution equation

$$\partial_p^{(n)} G_i^{(r)} = (\mathcal{Q}_{i+n-r}^p)_+ G_i^{(r)} - G_i^{(r)} (\mathcal{Q}_i^p)_+. \tag{29}$$

Note that in the case $r = n$, the latter becomes the Lax equation (28). Using standard arguments, one can show that equations (29) are properly defined individually. Indeed, taking into account (25), one can write $\mathcal{Q}_{i+n-r}^p = G_i^{(r)} \mathcal{Q}_i^p G_i^{(r)-1}$ or $\mathcal{Q}_{i+n-r}^p G_i^{(r)} = G_i^{(r)} \mathcal{Q}_i^p$ for any $p \in \boldsymbol{N}$. It follows from this that

$$(\mathcal{Q}_{i+n-r}^p)_+ G_i^{(r)} - G_i^{(r)} (\mathcal{Q}_i^p)_+ = G_i^{(r)} (\mathcal{Q}_i^p)_- - (\mathcal{Q}_{i+n-r}^p)_- G_i^{(r)}.$$

Thus the rhs of (29) as well as the lhs is a ΨDO of zero order. Moreover, in the case $r = 0$, i.e. when $G_i^{(0)} = H_i$, the rhs of (29) is a zeroth-order differential operator or simple function. It is now easy to establish $G_i^{(\ell,r)}$-evolutions with respect to KP flows. We have the following proposition.

Proposition 6. *By virtue of* (19) *and* (29)*, we have*

$$\partial_p^{(n)} G_i^{(\ell,r)} = (\mathcal{Q}_{i+\ell n-r}^p)_+ G_i^{(\ell,r)} - G_i^{(\ell,r)} (\mathcal{Q}_i^p)_+. \tag{30}$$

Proof. In the case $\ell = 0$, we obtain

$$\begin{aligned}
\partial_p^{(n)} G_i^{(0,r)} &= \partial_p^{(n)} (G_{i-n}^{(r)} H_{i-n}^{-1}) = \{(\mathcal{Q}_{i-r}^p)_+ G_{i-n}^{(r)} - G_{i-n}^{(r)} (\mathcal{Q}_{i-n}^p)_+\} H_{i-n}^{-1} \\
&\quad - G_{i-n}^{(r)} H_{i-n}^{-1} \{(\mathcal{Q}_i^p)_+ H_{i-n} - H_{i-n} (\mathcal{Q}_{i-n}^p)_+\} H_{i-n}^{-1} \\
&= (\mathcal{Q}_{i-r}^p)_+ G_i^{(0,r)} - G_i^{(0,r)} (\mathcal{Q}_i^p)_+.
\end{aligned}$$

Since $G_i^{(1,r)} = G_i^{(r)}$, equation (30) in the case $\ell = 1$ immediately follows from (29). The proof of (30) proceeds by induction. Assume that (30) is valid for some ℓ, then

$$\begin{aligned}
\partial_p^{(n)} G_i^{(\ell+1,r)} &= \partial_p^{(n)} (G_{i+n}^{(\ell,r)} H_i) = \{(\mathcal{Q}_{i+(\ell+1)n-r}^p)_+ G_{i+n}^{(\ell,r)} - G_{i+n}^{(\ell,r)} (\mathcal{Q}_{i+n}^p)_+\} H_i \\
&\quad + G_{i+n}^{(\ell,r)} \{(\mathcal{Q}_{i+n}^p)_+ H_i - H_i (\mathcal{Q}_i^p)_+\} \\
&= (\mathcal{Q}_{i+(\ell+1)n-r}^p)_+ G_i^{(\ell+1,r)} - G_i^{(\ell+1,r)} (\mathcal{Q}_i^p)_+.
\end{aligned}$$

This proves (30) for positive integers ℓ. By similar arguments equation (30) is shown for negative ℓ. □

Proposition 7. *Equations* (30) *are pair-wise compatible.*

Proof. One must show that permutation relation (24) is invariant with respect to KP flows. With the identities $\ell_1 + \ell_2 = \ell_3 + \ell_4$ and $r_1 + r_2 = r_3 + r_4$, we have

$$\begin{aligned}
\partial_p^{(n)}(G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2}G^{(\ell_2,r_2)}_i) &= \{(\mathcal{Q}^p_{i+(\ell_1+\ell_2)n-r_1-r_2})_+G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} - G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2}(\mathcal{Q}^p_{i+\ell_2 n-r_2})_+\}G^{(\ell_2,r_2)}_i \\
&\quad + G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2}\{(\mathcal{Q}^p_{i+\ell_2 n-r_2})_+G^{(\ell_2,r_2)}_i - G^{(\ell_2,r_2)}_i(\mathcal{Q}^p_i)_+\} \\
&= (\mathcal{Q}^p_{i+(\ell_1+\ell_2)n-r_1-r_2})_+G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2}G^{(\ell_2,r_2)}_i - G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2}G^{(\ell_2,r_2)}_i(\mathcal{Q}^p_i)_+ \\
&= (\mathcal{Q}^p_{i+(\ell_3+\ell_4)n-r_3-r_4})_+G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3}G^{(\ell_3,r_3)}_i - G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3}G^{(\ell_3,r_3)}_i(\mathcal{Q}^p_i)_+ \\
&= \{(\mathcal{Q}^p_{i+(\ell_3+\ell_4)n-r_3-r_4})_+G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} - G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3}(\mathcal{Q}^p_{i+\ell_3 n-r_3})_+\}G^{(\ell_3,r_3)}_i \\
&\quad + G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3}\{(\mathcal{Q}^p_{i+\ell_3 n-r_3})_+G^{(\ell_3,r_3)}_i - G^{(\ell_3,r_3)}_i(\mathcal{Q}^p_i)_+\} = \partial_p^{(n)}(G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3}G^{(\ell_3,r_3)}_i).
\end{aligned}$$

Therefore we have proved that equations (30) are pair-wise consistent. □

The fact that $\psi_i(t,z) = \hat{w}_i \mathrm{e}^{\xi(t^{(n)},z)}$ are KP wave eigenfunctions force them to be expressible via τ-functions

$$\psi_i(t,z) = \frac{\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)}-[z^{-1}],\ldots)}{\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)},\ldots)}\mathrm{e}^{\xi(t^{(n)},z)}$$

where $[z^{-1}] \equiv (1/z, 1/(2z^2),\ldots)$. Define $\Phi_i^{(n)} = \Phi_i^{(n)}(t)$ via $H_i\Phi_i^{(n)} = 0$, or equivalently through the following relation:

$$\partial^{(n)}\Phi_i^{(n)} = \Phi_i^{(n)}\sum_{s=1}^{n} a_0^{(n)}(i+s-1).$$

Taking into consideration (29), we get

$$\partial_p^{(n)}(H_i\Phi_i^{(n)}) = (\mathcal{Q}^p_{i+n})_+H_i\Phi_i^{(n)} - H_i(\mathcal{Q}^p_i)_+\Phi_i^{(n)} + H_i\partial_p^{(n)}\Phi_i^{(n)} = 0.$$

From this we derive $\partial_p^{(n)}\Phi_i^{(n)} = (\mathcal{Q}^p_i)_+\Phi_i^{(n)} + \alpha_i\Phi_i^{(n)}$ where α_i are some constants. Commutativity condition $\partial_p^{(n)}\partial_q^{(n)}\Phi_i^{(n)} = \partial_q^{(n)}\partial_p^{(n)}\Phi_i^{(n)}$ leads to evolution equations for KP eigenfunctions $\partial_p^{(n)}\Phi_i^{(n)} = (\mathcal{Q}^p_i)_+\Phi_i^{(n)}$, i.e. $\alpha_i = 0$. Thus the relations $\mathcal{Q}_{i+n} = H_i\mathcal{Q}_iH_i^{-1}$ defines DB transformations with eigenfunctions $\Phi_i^{(n)} = \tau_{i+n}^{(n)}/\tau_i^{(n)}$ [11]. Recall that an eigenfunction of the Lax operator $\mathcal{Q}$ contains information about DB transformation $\tau \to \overline{\tau} = \Phi\tau$ while the identity

$$\{\tau(t-[z^{-1}]),\overline{\tau}(t)\} + z(\tau(t-[z^{-1}])\overline{\tau}(t) - \overline{\tau}(t-[z^{-1}])\tau(t)) = 0$$

with $\{f,g\} \equiv \partial f\cdot g - \partial g\cdot f$ holds.

Proposition 8. *We have*

$$a_\ell^{(n,r)}(i) = \frac{p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau^{(n)}_{i-\ell n+r}\bullet\tau^{(n)}_i}{\tau^{(n)}_{i-\ell n+r}\tau^{(n)}_i}. \tag{31}$$

Proof. To show (31), we need the well known identity [8]

$$\mathrm{res}_z[(P\mathrm{e}^{xz})\cdot(Q\mathrm{e}^{-xz})] = \mathrm{res}_\partial PQ^* \tag{32}$$

where $P = \sum_{k\in Z} p_k(x)\partial^k$ and $Q = \sum_{k\in Z} q_k(x)\partial^k$ are two arbitrary ΨDOs and Q^* is the formal adjoint to Q.

To further use the identity (32) we set $P = G_i^{(\ell,r)}\hat{w}_i$ and $Q = \hat{w}_i^{*-1}$. Taking into account (20), (32) and applying proposition 2, we get

$$\begin{aligned}
a_\ell^{(n,r)}(i+\ell n-r) &= \mathrm{res}_z[(G_i^{(\ell,r)}\hat{w}_i \mathrm{e}^{x^{(n)}z})(\hat{w}_i^{*-1}\mathrm{e}^{-x^{(n)}z})] \\
&= \mathrm{res}_z[(G_i^{(\ell,r)}\hat{w}_i \mathrm{e}^{\xi(t^{(n)},z)})(\hat{w}_i^{*-1}\mathrm{e}^{-\xi(t^{(n)},z)})] \\
&= \mathrm{res}_z[(G_i^{(\ell,r)}\psi_i(t,z))\psi_i^*(t,z)] = \mathrm{res}_z[z^\ell \psi_{i+\ell n-r}(t,z)\psi_i^*(t,z)] \\
&= \mathrm{res}_z\left[z^\ell \frac{\tau_{i+\ell n-r}^{(n)}(t^{(1)},\ldots,t^{(n)}-[z^{-1}],\ldots)\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)}+[z^{-1}],\ldots)}{\tau_{i+\ell n-r}^{(n)}(t^{(1)},\ldots,t^{(n)},\ldots)\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)},\ldots)}\right] \\
&= \frac{1}{(\ell+1)!}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{\ell+1} \\
&\quad\times\left[\frac{\tau_{i+\ell n-r}^{(n)}(t^{(1)},\ldots,t^{(n)}-[u],\ldots)\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)}+[u],\ldots)}{\tau_{i+\ell n-r}^{(n)}(t^{(1)},\ldots,t^{(n)},\ldots)\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)},\ldots)}\right]\Bigg|_{u=0}.
\end{aligned}$$

Now using the technical identity (4) we obtain

$$a_\ell^{(n,r)}(i+\ell n-r) = \frac{p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau_i^{(n)} \bullet \tau_{i+\ell n-r}^{(n)}}{\tau_i^{(n)}\tau_{i+\ell n-r}^{(n)}}.$$

Shifting $i \to i-\ell n+r$ in the latter we arrive at (31). □

Remark 4. By (31), we can express Q^r in terms of τ-functions as

$$Q^r = \sum_{\ell=0}^{\infty} \mathrm{diag}\left(\frac{p_\ell(\tilde{D}_{t^{(n)}})\tau_{i-(\ell-1)n+r}^{(n)} \bullet \tau_i^{(n)}}{\tau_{i-(\ell-1)n+r}^{(n)}\tau_i^{(n)}}\right)_{i\in Z} z^{\ell(n-1)}\Lambda^{r-\ell n}. \tag{33}$$

In the case of ordinary discrete KP hierarchy ($n=1$), (33) coincides with the formula (0.13) of the paper [9].

Since $a_\ell^{(n,0)}(i) = 0$, then as a consequence of the above proposition we deduce the following bilinear equations:

$$p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau_{i-\ell n}^{(n)} \bullet \tau_i^{(n)} = 0 \qquad \ell = 0, 1, \ldots. \tag{34}$$

With the well known bilinear identity for KP wave eigenfunction (see, for example, [13])

$$\mathrm{res}_z[(\partial_1^{k_1}\ldots\partial_m^{k_m}\psi(t,z))\cdot\psi^*(t,z)] = 0$$

and the fact that $G_i^{(\ell,\ell n)}$, $\ell = 0, 1, \ldots$ are purely differential operators, one can deduce the following proposition.

Proposition 9. *The τ-functions of the nth discrete KP hierarchy satisfy*

$$\begin{aligned}
&\mathrm{res}_z[z^\ell \tau_{i+\ell n}^{(n)}(t^{(1)},\ldots,t^{(n)}-[z^{-1}],\ldots)\tau_i^{(n)}(t^{(1)},\ldots,t^{(n)'}+[z^{-1}],\ldots) \\
&\qquad\times \exp\xi(t^{(n)}-t^{(n)'},z)] = 0 \qquad \forall t^{(n)}, t^{(n)'}, \ell = 0, 1, 2\ldots.
\end{aligned} \tag{35}$$

Relation (35) simply means that $\tau_i^{(n)}$ and $\tau_{i+n}^{(n)}$, for each $i \in \boldsymbol{Z}$ are related by DB transformation with corresponding eigenfunction $\Phi_i^{(n)} = \tau_{i+n}^{(n)}/\tau_i^{(n)}$, as was mentioned before. Of course, in the case $n=1$, (35) coincides with the known bilinear identity for the discrete KP τ-function [9]. One can check that relations (34) constitute a part of all bilinear equations coded in (35).

Unfortunately, at present we do not know how to construct the full set of bilinear identities characterizing the extended discrete KP hierarchy and we leave this question for future investigations.

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